Practices before the class (March 29)

Let $\mathbb{P}_2$ be the vector space of the polynomials of degree at most 2. Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^4$ defined by $T(\mathbf{p}(t)) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \\ p'(2) \end{bmatrix}$.

Find a basis for the range of T.

Practices before the class (March 29)

Answer:

- Any element in $\mathbb{P}_2$ can be written as $p(t) = at^2 + bt + c$. Note $p'(t) = 2at + b$.
- Then $T(at^2 + bt + c) = \begin{bmatrix} a \cdot 0^2 + b \cdot 0 + c \\ a \cdot 1^2 + b \cdot 1 + c \\ a \cdot 2^2 + b \cdot 2 + c \\ 2a \cdot 2 + b \end{bmatrix} = \begin{bmatrix} c \\ a + b + c \\ 4a + 2b + c \\ 4a + b \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 4 \\ 4 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$
  for any $p(t) = at^2 + bt + c$ in $\mathbb{P}_2$.
- This means any element in the range of T can be written as a linear combination of $\begin{bmatrix} 0 \\ 1 \\ 4 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. We can also check that they are linearly independent.
- A basis for the range of T is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

5.7 Applications to Differential Equations

Consider a system of **differential equations**:

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

We can write the system as a matrix differential equation

$$\mathbf{x}'(t) = A\mathbf{x}(t) \tag{1}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

A solution of **equation** (1) is a vector-valued function that satisfies (1) for all t in some interval of real numbers, such as $t \geq 0$.

Remark:

1. **Superposition of Solutions.** If $\mathbf{u}$ and $\mathbf{v}$ are solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$, then $c\mathbf{u} + d\mathbf{v}$ is also a solution.

   We have $\underline{\vec{u}' = A\vec{u}}$, $\underline{\vec{v}' = A\vec{v}}$ since $\vec{u}, \vec{v}$ are solutions to $\vec{x}'(t) = A\vec{x}(t)$.

   We check $c\vec{u} + d\vec{v}$ satisfies $\vec{x}'(t) = A\vec{x}(t)$:

   $$(c\vec{u} + d\vec{v})' = c\vec{u}' + d\vec{v}' = cA\vec{u} + dA\vec{v} = A(c\vec{u} + d\vec{v})$$

2. **Fundamental Set of Solutions.** If A is $n \times n$, then there are n linearly independent functions in a fundamental set and each solution of (1) is a unique linear combination of these n functions.

3. **Initial Value Problem.** If a vector $\mathbf{x}_0$ is specified, then the initial value problem is to find the unique function $\mathbf{x}$ such that

$$\begin{aligned} \mathbf{x}'(t) &= A\mathbf{x}(t) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

Example 1. Consider $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Here the matrix A is diagonal, we call the system **decoupled**. Find solutions to this system.

ANS: We have $\begin{cases} x_1'(t) = 3x_1(t) \\ x_2'(t) = -5x_2(t) \end{cases}$ and notice each function only depends on itself (decoupled).

$$x_1' = \frac{dx_1}{dt} = 3x_1$$

$$\Rightarrow \frac{dx_1}{x_1} = 3\,dt \quad \left(\text{multiply } \frac{dt}{x_1} \text{ both sides}\right)$$

$$\Rightarrow \int \frac{dx_1}{x_1} = 3\int dt \quad (\text{take integral both sides}).$$

$$\Rightarrow \ln|x_1| = 3t + C \quad \left(\text{Recall } \int \frac{dx}{x} = \ln|x| + c \quad \int dt = t + c\right)$$

$$\Rightarrow e^{\ln|x_1|} = e^{3t+C} \quad (\text{Take exp both sides}).$$

$$\Rightarrow x_1 = \underbrace{\pm e^{C}}\cdot e^{3t} \longrightarrow \text{again it is a constant. let it to be } C_1$$

$$\Rightarrow x_1(t) = C_1 e^{3t} \text{ for any } C_1 \text{ is a solution to } x_1(t).$$

Similarly, $x_2(t) = C_2 e^{-5t}$ is a solution to the second equation.

Thus

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} C_1 e^{3t} \\ C_2 e^{-5t} \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t} \quad (2)$$

for any constant C_1 and C_2.

We call Eq (2) the general solution for the given system.

The example suggests a solution might be in the form of $\vec{x}(t) = \vec{v}e^{\lambda t}$, for some λ and a nonzero vector $\vec{v}$.

Remark: The Eigenvalue Method for Solving $\mathbf{x}'(t) = A\mathbf{x}(t)$

- We plug $\vec{x}(t) = \vec{v}e^{\lambda t}$ into $\vec{x}'(t) = A\vec{x}(t)$.

$$\vec{x}'(t) = \underline{\vec{v}\lambda \cancel{e^{\lambda t}}} = A\vec{x}(t) = \underline{A\vec{v}\cancel{e^{\lambda t}}}$$

$$\Rightarrow \quad A\vec{v} = \lambda\vec{v}$$

Thus λ is an eigenvalue for A and $\vec{v}$ is the corresponding eigenvector.

- Therefore, to solve $\vec{x}' = A\vec{x}$, we can start from finding eigenvalues and eigenvectors for A.

We summarize the method in Example 2 & Example 4 as follows:

Constant Coeff. Homogeneous System:

Constant Coeff. Homogeneous: $\mathbf{x}' = A\mathbf{x}$

Solution: $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots$,
where $\mathbf{x}_i$ are fundamental solutions from eigenvalues & eigenvectors.
The method is described as below.

The Eigenvalue Method for $\mathbf{x}' = A\mathbf{x}$ in this section:

We consider A to be 2×2, then the general solution is $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$, with the fundamental solutions $\mathbf{x}_1(t), \mathbf{x}_2(t)$ found has follows.

- Distinct Real Eigenvalues. $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}, \mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}$
- Complex Eigenvalues. $\lambda_{1,2} = p \pm qi$. (*suggestion: use an example to remember the method*)

  If $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ is an eigenvector associated with $\lambda = p + qi$, then

  $\mathbf{x}_1(t) = e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt)$, $\mathbf{x}_2(t) = e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt)$.

Example 2. The circuit in **Figure 1** can be described by the differential equation

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -(1/R_1 + 1/R_2)/C_1 & 1/(R_2C_1) \\ 1/(R_2C_2) & -1/(R_2C_2) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad (3)$$

where $x_1(t)$ and $x_2(t)$ are the voltages across the two capacitors at time t. Suppose resistor R_1 is 1 ohm, R_2 is 2 ohms, capacitor C_1 is 1 farad, and C_2 is .5 farad, and suppose there is an initial charge of 5 volts on capacitor C_1 and 4 volts on capacitor C_2. Find formulas for $x_1(t)$ and $x_2(t)$ that describe how the voltages change over time.

ANS: $R_1 = 1$, $R_2 = 2$, $x_1(0) = 5$

$C_1 = 1$, $C_2 = 0.5$, $x_2(0) = 4$.

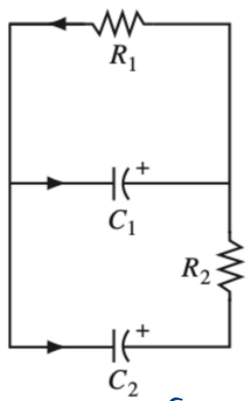


FIGURE 1

Let A be the 2×2 matrix in (3)

Then

$$A = \begin{bmatrix} -(1/1 + 1/2)\cdot 1 & 1/2 \\ 1/2\times0.5 & -1/2\times0.5 \end{bmatrix}$$

So we need to solve the initial value problem:

$$\boxed{\vec{x}' = \begin{bmatrix} -1.5 & 0.5 \\ 1 & -1 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}}$$

From the discussion above, we need to find solutions in the form $\vec{v}e^{\lambda t}$, where λ is an eigenvalue for A and $\vec{v}$ is the corresponding eigenvector.

$$|A - \lambda I| = \begin{vmatrix} -1.5-\lambda & 0.5 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda+1.5) - 0.5 = \lambda^2 + 2.5\lambda + 1 = 0$$

$$\Rightarrow (\lambda+0.5)(\lambda+2) = 0 \Rightarrow \lambda_1 = -0.5 \text{ and } \lambda_2 = -2.$$

For $\lambda_1 = -0.5$, the eigenvector is $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $\lambda_2 = -2$, the eigenvector is $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Thus $\vec{x}_1 = \vec{v}_1 e^{\lambda_1 t}$

$$\vec{x}_2 = \vec{v}_2 e^{\lambda_2 t}$$

both satisfy $\vec{x}\,' = A\vec{x}$. Moreover, they are linearly independent.

Thus a general solution is their linear combination.

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

$$\Rightarrow \vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$ for any constants c_1 and c_2

Note $\vec{x}(t)$ also needs to satisfy the initial value:

$$\vec{x}(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

i.e. $$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \overset{1}{\cancel{e^{-0.5\cdot 0}}} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \overset{1}{\cancel{e^{-2\cdot 0}}} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\Rightarrow c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{cases} c_1 = 3 \\ c_2 = -2. \end{cases}$$

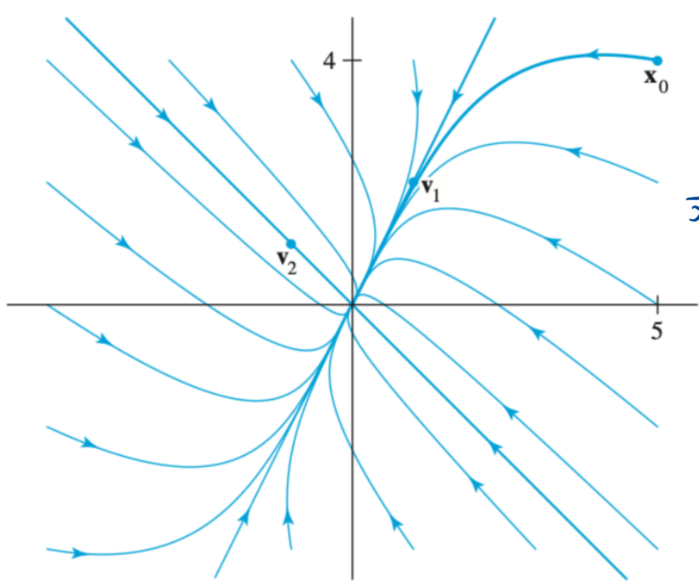


FIGURE 2 The origin as an attractor.

So the solution to $\vec{x}\,' = A\vec{x}$, $\vec{x}(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ is

$$\vec{x}(t) = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.5t} - 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

i.e. $$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3e^{-0.5t} + 2e^{-2t} \\ 6e^{-0.5t} - 2e^{-2t} \end{bmatrix}$$

Decoupling a Dynamical System

Let A be $n \times n$ and has n linearly independent eigenvectors, i.e., A is diagonalizable.

We use **Example 3** to explain how to decouple the equation $\mathbf{x}' = A\mathbf{x}$. For a general discussion about the process, please refer to Page 324-325 in our textbook.

Example 3. Let $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$.

Make a change of variable that decouples the equation $\mathbf{x}' = A\mathbf{x}$. Write the equation $\mathbf{x}(t) = P\mathbf{y}(t)$ and show the calculation that leads to the uncoupled system $\mathbf{y}' = D\mathbf{y}$, specifying P and D.

ANS: We can compute the eigenvalues and eigenvectors for A.

- $\lambda_1 = -2 \qquad \vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
- $\lambda_2 = -1 \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- To decouple $\vec{x}' = A\vec{x}$. set $P = [\vec{v}_1 \ \ \vec{v}_2] = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$.

and $D = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$. Then $A = PDP^{-1}$ and $D = P^{-1}AP$

- Substitute $\vec{x}(t) = P\vec{y}(t)$ into $\vec{x}' = A\vec{x}$, we have

$$\vec{x}'(t) = (P\vec{y}(t))' = P\vec{y}'(t)$$

$$= AP\vec{y}(t) = PD\overset{I}{\cancel{P^{-1}P}}\vec{y}(t)$$

$$\Rightarrow P^{-1}P\vec{y}'(t) = P^{-1}PD\vec{y}(t)$$

$$\Rightarrow \vec{y}'(t) = D\vec{y}(t)$$

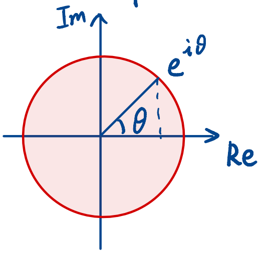
$$\text{or} \quad \vec{y}\,' = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \vec{y}$$

$$\Rightarrow \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The next formula is useful if we have complex eigenvalues when solving $\vec{x}\,' = A\vec{x}$

Euler's formula for complex numbers:

- Euler's formula : $e^{i\theta} = \cos\theta + i\sin\theta$



- $e^{z} = e^{x+iy} = e^{x} \cdot e^{iy} = e^{x}(\cos y + i\sin y)$

  where $z = x+iy$ is any complex number.

Complex Eigenvalues

In **Example 4**, a real matrix A has a pair of complex eigenvalues λ and $\bar{\lambda}$, with associated complex eigenvectors $\mathbf{v}$ and $\overline{\mathbf{v}}$. So two solutions of $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{x}_1(t) = \mathbf{v}e^{\lambda t} \quad \text{and} \quad \mathbf{x}_2(t) = \overline{\mathbf{v}}e^{\bar{\lambda}t},$$

which are functions in terms of complex numbers 😕.

In practice, we want to find **real-valued solutions** 🧐.

We use this example to explain how to find real-valued solutions for $\mathbf{x}' = A\mathbf{x}$ in such cases.

Example 4. Find the solution to the initial value problem $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} -2 & -2.5 \\ 10 & -2 \end{bmatrix}$ and $\mathbf{x}_0 = \mathbf{x}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

Note: You can use the following online calculator to graph the solution curve:

https://aeb019.hosted.uark.edu/pplane.html

ANS: $|A-\lambda I| = \begin{vmatrix} -2-\lambda & -2.5 \\ 10 & -2-\lambda \end{vmatrix} = (\lambda+2)^2 + 25 = 0$

$\Rightarrow \lambda + 2 = \pm 5i \Rightarrow \lambda = -2 \pm 5i$

For $\lambda_1 = -2+5i$, we solve $(A-\lambda I)\vec{v} = \vec{0}$, the augmented matrix is

$$\begin{bmatrix} -5i & -2.5 & 0 \\ 10 & -5i & 0 \end{bmatrix}$$

Notice $R1 \times 2i = R2$. thus $R1$ and $R2$ give the same equation.

$$10x_1 - 5i\,x_2 = 0$$

$$\Rightarrow 2x_1 = i x_2$$

$$\Rightarrow \vec{v}_1 = 2 \times \begin{bmatrix} \frac{i}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} i \\ 2 \end{bmatrix} \text{ is an eigenvector for } \lambda_1 = -2+5i.$$

By §5.5. we know $\vec{v}_2 = \overline{\vec{v}_1} = \begin{bmatrix} -i \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda_2 = \overline{\lambda_1} = -2-5i$.

Thus $\vec{x}(t) = c_1 \begin{bmatrix} i \\ 2 \end{bmatrix} e^{(-2+5i)t} + c_2 \begin{bmatrix} -i \\ 2 \end{bmatrix} e^{(-2-5i)t}$ is a complex-valued solution.

However, we often want to find real-valued solutions.

To do this, we know

$$\begin{bmatrix} i \\ 2 \end{bmatrix} e^{(-2+5i)t} \quad \text{is a solution.}$$

Rewrite it as

$$\vec{x}(t) = \begin{bmatrix} i \\ 2 \end{bmatrix} e^{(-2+5i)t}$$

$$= \begin{bmatrix} i \\ 2 \end{bmatrix} e^{-2t}(\cos 5t + i \sin 5t) \quad \left(\text{as } e^{(x+iy)t} = e^{xt}(\cos yt + i \sin yt)\right)$$

$$= \begin{bmatrix} i e^{-2t}\cos 5t - e^{-2t}\sin 5t \\ 2e^{-2t}\cos 5t + 2i e^{-2t}\sin 5t \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -e^{-2t}\sin 5t \\ 2e^{-2t}\cos 5t \end{bmatrix}}_{Re(\vec{x}(t))} + i \underbrace{\begin{bmatrix} e^{-2t}\cos 5t \\ 2e^{-2t}\sin 5t \end{bmatrix}}_{Im(\vec{x}(t))}$$

Note $Re(\vec{x}(t))$ and $Im(\vec{x}(t))$ are both solutions to $\vec{x}' = A\vec{x}$.

Moreover, they are linearly independent. Thus a general solution (real-valued) can be a linear combination of them.

$$\vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} -\sin t \\ 2\cos t \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} \cos t \\ 2\sin t \end{bmatrix}$$

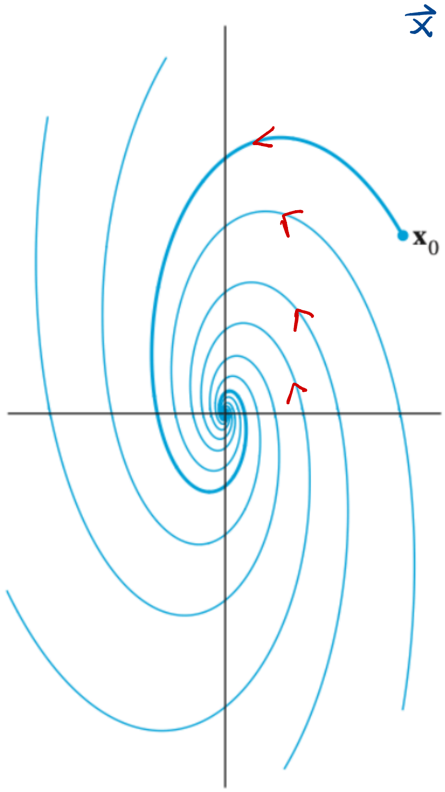


FIGURE 5
The origin as a spiral point.

The initial condition $\vec{x}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ gives

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} c_1 = 1.5 \\ c_2 = 3 \end{cases}$$

Thus $\vec{x}(t) = 1.5\, e^{-2t} \begin{bmatrix} -\sin t \\ 2\cos t \end{bmatrix} + 3\, e^{-2t} \begin{bmatrix} \cos t \\ 2\sin t \end{bmatrix}$

Note $\lim_{t \to \infty} \vec{x}(t) = \vec{0}$

Summary 1: Solving $\mathbf{x}' = A\mathbf{x}$ when A has complex eigenvalues

We summarize the general method described in **Example 4** below:

Assume we have complex eigenvalues $\lambda = p + qi, \bar{\lambda} = p - qi$.

If $\mathbf{v}$ is an eigenvector associated with $\lambda = p + qi$, then $\mathbf{v}$ can be written as $\mathbf{v} = \mathbf{a} + i\mathbf{b}$.

Then we have the solution

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t} = (\mathbf{a} + i\mathbf{b})e^{(p+qi)t}$$

$$\Rightarrow \quad \mathbf{x}(t) = e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt) + ie^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt)$$

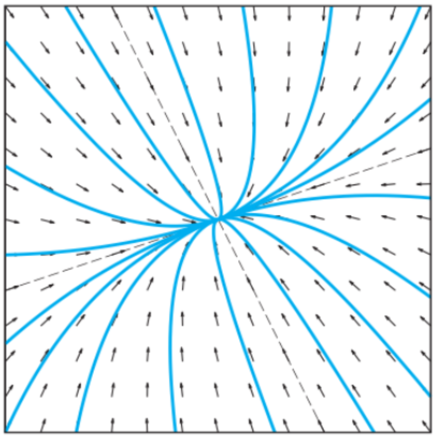
Then we get the real-valued solutions

$$\begin{cases} \mathbf{x}_1(t) = \mathrm{Re}(\mathbf{x}(t)) = e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt) \\ \mathbf{x}_2(t) = \mathrm{Im}\left(\mathbf{x}(t)\right) = e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt) \end{cases}$$

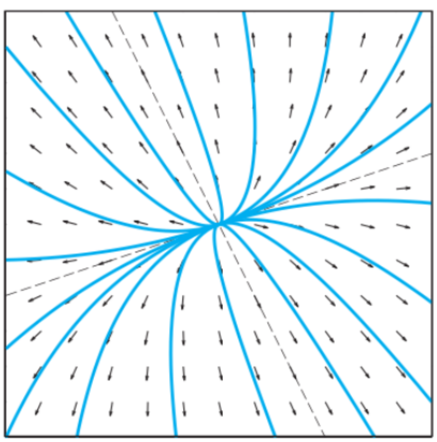
Summary 2: Gallery of Typical Solution Graphs (Trajectories) for the System $\mathbf{x}' = A\mathbf{x}$

We summarize the typical trajectories that show up in this section:

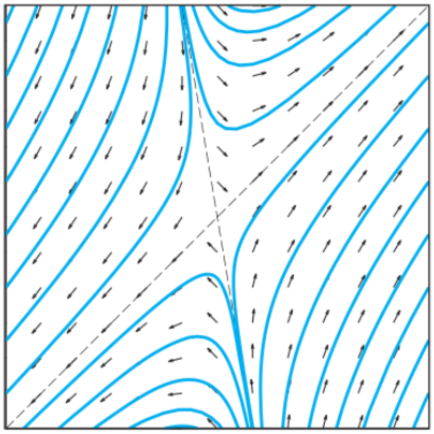
1. The origin is an **attractor** (or **sink**)
   - This happens when A has **distinct negative real eigenvalues**.
   - The arrows are pointing towards the origin.
   - Check **Example 2** for details.



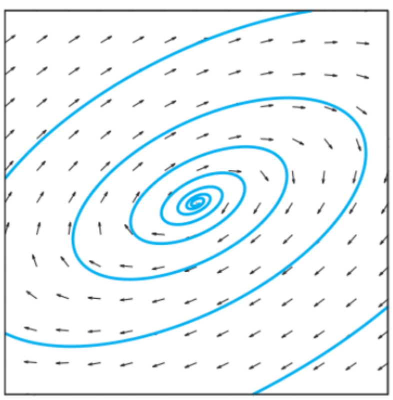
2. The origin is a **repeller** (or **source**)
   - This happens when A has **distinct positive real eigenvalues**.
   - The arrows are traversed away from the origin.



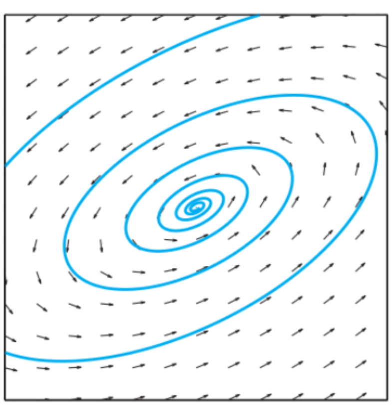
3. The origin is a **saddle point**.
    - This happens when A has **real eigenvalues of opposite sign.**
    - Check **Exercise 5** for details about the eigenvectors, greatest attraction, and greatest repulsion.



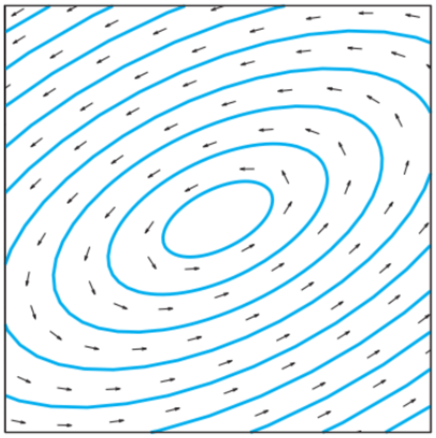
4. The origin is a **spiral point**.
    - This happens when A has **complex conjugate eigenvalues with nonzero real parts.**
    - If the eigenvalues have positive real parts, the trajectories spiral outward.



    - If the eigenvalues have negative real parts, the trajectories spiral inward. Check **Example 4.**



5. The origin is a **center** and the trajectories are ellipses about the origin.
    - This happens when A has purely imaginary eigenvalues.
    - Your **Handwritten Homework 28** is an example of this case.



Exercise 5. (The case when the origin is a saddle point)

Solve the initial value problem $\mathbf{x}'(t) = A\mathbf{x}(t)$ for $t \geq 0$, with $\mathbf{x}(0) = (3, 2)$. Classify the nature of the origin as an attractor, repeller, or saddle point of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. Find the directions of greatest attraction and/or repulsion. When the origin is a saddle point, sketch typical trajectories.

$$A = \begin{bmatrix} -2 & -5 \\ 1 & 4 \end{bmatrix}$$

Solution. $A = \begin{bmatrix} -2 & -5 \\ 1 & 4 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0.$

Eigenvalues: -1 and 3 .

$\underline{\text{For } \lambda = 3:}$ $\begin{bmatrix} -5 & -5 & 0 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -x_2$ with x_2 free. Take $x_2 = 1$ and $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\underline{\text{For } \lambda = -1:}$ $\begin{bmatrix} -1 & -5 & 0 \\ 1 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -5x_2$ with x_2 free. Take $x_2 = 1$ and $\mathbf{v}_2 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$.

The general solution of $\mathbf{x}' = A\mathbf{x}$ has the form $\mathbf{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix} e^{-t}$.

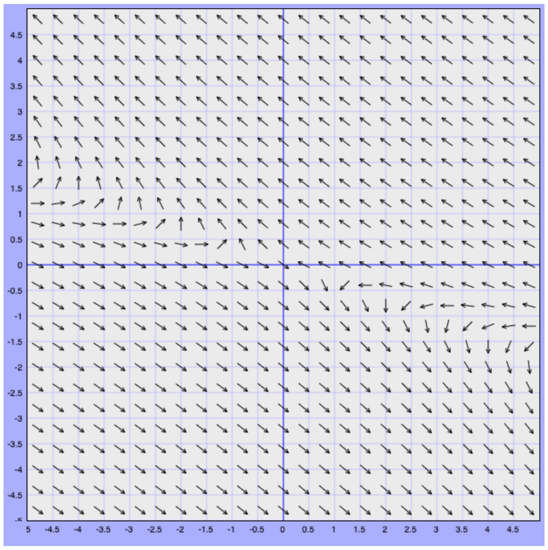
For the initial condition $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find c_1 and c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}(0)$:

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{x}(0)] = \begin{bmatrix} -1 & -5 & 3 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 13/4 \\ 0 & 1 & -5/4 \end{bmatrix}.$$

Thus $c_1 = 13/4$, $c_2 = -5/4$, and $\mathbf{x}(t) = \frac{13}{4} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t} - \frac{5}{4} \begin{bmatrix} -5 \\ 1 \end{bmatrix} e^{-t}$.

Since one eigenvalue is positive and the other is negative, the origin is a saddle point of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. The direction of greatest attraction is the line through $\mathbf{v}_2$ and the origin. The direction of greatest repulsion is the line through $\mathbf{v}_1$ and the origin.

The following diagram is obtained from the website:https://aeb019.hosted.uark.edu/pplane.html



Exercise 6. Construct the general solution of $\mathbf{x}' = A\mathbf{x}$ involving complex eigenfunctions and then obtain the general real solution. Describe the shapes of typical trajectories.

$$A = \begin{bmatrix} -6 & -11 & 16 \\ 2 & 5 & -4 \\ -4 & -5 & 10 \end{bmatrix}$$

Solution. We first find the eigenvalues for A by solving $|A - \lambda I| = 0$. The eigenvalues are $4, 3$ and 2.

By solving the equations $(A - \lambda I)\mathbf{x} = \mathbf{0}$, we find the eigenvector associated to $\lambda_1 = 4$ is $\mathbf{v}_1 = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$.

For $\lambda_2 = 3$, we have $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

For $\lambda_3 = 2$, we have $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Hence the general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{2t}$. The origin is a repeller, because all eigenvalues are positive. All trajectories tend away from the origin.

Exercise 7. Construct the general solution of $\mathbf{x}' = A\mathbf{x}$ involving complex eigenfunctions and then obtain the general real solution. Describe the shapes of typical trajectories.

$$A = \begin{bmatrix} 53 & -30 & -2 \\ 90 & -52 & -3 \\ 20 & -10 & 2 \end{bmatrix}$$

Solution. We first find the eigenvalues for A by solving $|A - \lambda I| = 0$. The eigenvalues are $5 + 2i, 5 - 2i$ and 1.

For $\lambda_1 = 5 + 2i$, we have $\mathbf{v}_1 = \begin{bmatrix} 23 - 34i \\ -9 + 14i \\ 3 \end{bmatrix}$.

For $\lambda_2 = 5 - 2i$, we have $\mathbf{v}_2 = \begin{bmatrix} 23 + 34i \\ -9 - 14i \\ 3 \end{bmatrix}$.

For $\lambda_3 = 1$, we have $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$.

Thus the general complex solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 23 - 34i \\ -9 + 14i \\ 3 \end{bmatrix} e^{(5+2i)t} + c_2 \begin{bmatrix} 23 + 34i \\ -9 - 14i \\ 3 \end{bmatrix} e^{(5-2i)t} + c_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^t$.

Rewriting the first eigenfunction yields

$$\begin{bmatrix} 23 - 34i \\ -9 + 14i \\ 3 \end{bmatrix} e^{5t}(\cos 2t + i \sin 2t) = \begin{bmatrix} 23\cos 2t + 34 \sin 2t \\ -9 \cos 2t - 14 \sin 2t \\ 3 \cos 2t \end{bmatrix} e^{5t} + i \begin{bmatrix} 23 \sin 2t - 34 \cos 2t \\ -9 \sin 2t + 14 \cos 2t \\ 3 \sin 2t \end{bmatrix} e^{5t}$$

Hence the general real solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 23\cos 2t + 34 \sin 2t \\ -9 \cos 2t - 14 \sin 2t \\ 3 \cos 2t \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 23 \sin 2t - 34 \cos 2t \\ -9 \sin 2t + 14 \cos 2t \\ 3 \sin 2t \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^t,$$

where c_1, c_2, and c_3 are real. The origin is a repeller, because the real parts of all eigenvalues are positive. All trajectories spiral away from the origin.